MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN0. Introduction to Logic.

Mathematical theory, at least in this course, is stated in the language of first-order logic. As such, we'll do an introduction of propositional logic – the basis machinery of first-order logic. Please do note the following discussion mostly covers how to use the concepts that are important to this course, particularly how to use implications and bijections. This is not meant to be a deeper dive on logic theory and so most of the terms written here are simplified.

Definition 0.1.

A **truth function** is a function that accepts truth values (either true or false) as input and produces a unique truth value as output. A **truth table** is a representation of a truth function that considers all possible input truth values and specifies the output truth value for each. A **logical operator/connective** is a symbol that represents a truth function.

Definition 0.2.

Let A and B represent truth values. The logical operators **NOT**, **OR**, **AND** are truth functions described by truth tables provided below. Note that the **NOT** operator accepts a single truth value; the **OR** and **AND** operators accept two.

A	**NOT** A
T	F
F	T

A	B	A **OR** B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	A **AND** B
T	T	T
T	F	F
F	T	F
F	F	F

The logical operator **NOT** is usually called negation; **AND** called conjuction; and **OR** called disjunction.

Observe that both **AND** and **OR** are commutative operators, e.g. A **AND** B is the same as saying B **AND** A. We can conclude this by comparing the truth tables of A **AND** B and B **AND** A and concluding they are equal as functions (i.e. same inputs generate same outputs).

Observe that the naming of these logical operators mostly match their respective English interpretations. For example, the **NOT** operator does as expected: turns true values into false and false values into true. However, we need to be careful that we don't confuse the meaning of a logical operator from the English meaning of its name. For example, the word **or** may sometimes indicate a choice between two things and that both things can't be chosen simultaneously. This is not the true of the logical operator **OR** which returns true even if both input values are true.

Definition 0.3. Implications.

Let A and B represent two truth values. An **implication** $A \Rightarrow B$ (read as A implies B) is a logical operator that takes in two truth values and is described by the following truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

There are a number of ways we indicate an implication in English. Let A and B represent two truth values.

The following statements often represent the relationship $A \Rightarrow B$:

1. $B \Leftarrow A$.
2. If A, then B.
3. B if A.
4. Let A be true. Then, B.
5. A is a **sufficient** condition for B.

Observe that for $A \Rightarrow B$: when the first argument A is false, the implication returns true regardless of the truth value of B. Only when the first argument A is true does the truth value of the implication signify the truth value of the second argument B.

Theorem 0.4. Negation of an Implication

Let A and B represent truth values. Then, the implication $A \Rightarrow B$ is equivalent to the function B **OR** (**NOT** A) and the negation of an implication **NOT** $\big(A \Rightarrow B\big)$ is $\big($**NOT** $B\big)$ **AND** A.

Observe that the negation of an implication is not another implication. Instead, it's a conjuction of two statements. If you look at higher level math, you might see this being used for proofs by contradiction. This result is useful if we know that an implication is false.

Definition 0.5. Converse and Contrapositive.

Let A and B represent two truth values. Let $A \Rightarrow B$ be an implication. The **converse** of $A \Rightarrow B$ is the relationship $B \Rightarrow A$. The **contrapositive** of $A \Rightarrow B$ is the relationship **NOT** $B \Rightarrow$ **NOT** A.

Theorem 0.6. Relating the converse and the contrapositive to the implication.

Let $A \Rightarrow B$ be an implication. Then, the converse $B \Rightarrow A$ is not the same function as $A \Rightarrow B$. However, the contrapositive **NOT** $B \Rightarrow$ **NOT** A is the same function. The truth tables below summarize the results.

A	B	$A \Rightarrow B$	$B \Rightarrow A$	**NOT** A	**NOT** B	**NOT** $B \Rightarrow$ **NOT** A
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

The main takeaway of the last statement should be that (1) the truth of an implication does not imply (we use imply logically here) the truth of its converse and (2) the truth of an implication implies the truth of its contrapositive. Therefore, in places where the contrapositive form of an implication is more useful, we can use the contrapositive. However, the same doesn't apply for the converse.

Additionally, there are some statements that seem like they signify an implication statement when they actually state a converse relationship (relative to the seeming implication). That is, let A and B be statements. Then, the following statements represent the relationship $B \Rightarrow A$ (the converse of $A \Rightarrow B$):

1. A only if B.
2. A is a **necessary** condition for B.

Be careful with the first statement. This phrasing seems unintuitive. We'll try not to use it in this course but you may see in other resources/references.

Definition 0.7. Bijection.

Let A and B represent two truth values. A **bijection** $A \Leftrightarrow B$ (read as A if and only if B) is the function given by $(A \Rightarrow B)$ **AND** $(B \Rightarrow A)$. We say that statements A and B are equivalent when there exists a bijection between them (i.e. they are equal as truth functions).

We've added the truth table for $A \Leftrightarrow B$ below:

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Observe that given a bijection $A \Leftrightarrow B$ is true: the truth value of A determines the truth value of B and the truth value of B determines the truth value of A. This is why we often refer to a logical bijection as an equivalence.

Later on, we'll be (in some sense) operating at the level of outputs. That is, we assume that a truth function evaluates true and we also assume the truth of some of the inputs. Then, we can conclude the truth values of the other inputs. The logical rules that let us make these conclusions are called **argument forms** or **inference rules**. We've talked about these in the previous paragraphs but we'll state some of these below for completion.

Theorem 0.8. Argument Forms

An **argument form** is a conclusion of some truth value based on the truth of other statements. Let A and B be truth values. The following are common argument forms:

1. **Simplification.** Assume A **AND** B evaluates true. Then, A is true.
   (By symmetry, we can also conclude that B is true.)
2. **Disjuntive Syllogism.** Assume A **OR** B is true. Assume A is false. Then, B must be true.
   (By symmetry, assuming B is false, we can conclude that A must be true)
3. **Modus Ponens.** Assume $A \Rightarrow B$ evaluates true and assume A is true. Then, B must be true.
4. **Modus Tollens.** Assume $A \Rightarrow B$ evaluates true and assume B is false. Then, A must be false.
   (Observe that this is modus ponens on the contrapositive)

For a bigger list of argument forms, please visit Wikipedia: Propositional Calculus: Basic and Derived Argument Forms.

While knowing the names of these argument forms may help us talk about them (which is why they're included here), their names are most of the time not invoked in explanations. In fact, when we say something is a logical conclusion, some of these argument forms are used in the background.

Observe we can verify the correctness of these argument forms by looking at the relevant truth tables. Take, for example, modus ponens. Let A and B take truth values. Assuming $A \Rightarrow B$ evaluates true, there are only three possibilities of A and B: (1) A is true and B is true; (2) A is false and B is true; and (3) A is false and B is false. With the additional assumption that A is true, the only possibility is the first one: A is true and B is true. Observe that if we assume B is true, possibilities (1) and (2) are still valid and therefore, we can't narrow down the truth value of A. This is often what we refer to when we say the truth of an implication doesn't imply the truth of its converse.

Similarly, for disjunctive syllogism: Assuming that A **OR** B is true limits the possibilities to three: (1) A is true and B is true; (2) A is true and B is false; and (3) A is false and B is true. With the additional assumption that A is false, we're limited to possibility (3) and therefore, B must be true.

Finally, observe that the previous discussion does not talk about the statements A and B being functions of objects, i.e. the statements are fixed and therefore, the object of concern is also fixed. Often, in mathematics, we consider a certain set of objects and determine whether said object has a certain property. To account for this, we introduce first-order logic.

Definition 0.9. Predicates.

Let X be some space of objects. A **predicate** $P(x)$ is a function that accepts objects $x \in X$ and returns a truth value. Often, writing $P(x)$ denotes that $P(x)$ evaluates to true for the object x.

Since predicates have truth values are outputs, everything else before this (e.g. negation, implication, bijection) also applies to predicates.

The notion of a predicate is important since we often don't consider the "universal" set for our results (whichever counts as universal). For example, when examining functions $f(x)$, we only consider x such that x is in the domain of $f(x)$, not all real numbers x. In the case of $f(x) = \log(x)$, the domain is the set of all x such that $x > 0$. Later on, we'll consider more sets of objects such as the following:

1. $\mathbb{R}^n$: the set of all n-tuples with coefficients in $\mathbb{R}$.
2. $\mathbb{R}^{n \times m}$: the set of all $n \times m$ matrices with real number entries.
3. $\mathbb{C}^{n \times m}$: the set of all $n \times m$ matrices with complex number entries.
4. $\mathbb{R}^{\mathbb{N}}$: the set of all sequences with real number entries (seen in MTH 265).

Most results only apply to certain sets of objects.

Definition 0.10. Quantifiers.

A **quantifier** is a statement on some set of objects X and how they relate to some predicate $P(x)$. A quantifier also has a truth value.

A **universal quantifier** is a statement that says for any object $x \in X$, the predicate $P(x)$ evaluates true. This relationship is often denoted as $\forall\, x \in X, P(x)$ read as "for all x in X, $P(x)$ is true".

A **existential quantifier** is a statement that says for at least one object in $a \in X$, the predicate $P(a)$ evaluates true. This relationship is often denoted as $\exists\, a \in X, P(a)$ read as "there exists a in X such that $P(a)$ is true".

The theorems we'll see and use later in this course are often stated as universal or existential quantifiers. In fact, some of the theorems and definitions discussed in algebra and calculus are universal and existential quantifiers. We list some of those examples below.

Example 0.10.1. The polynomial $f(x) = x^3 - 3x + 4$ has a zero.

The definition of a zero of a function $f(x)$ is a value x such that $f(x) = 0$. This is our predicate. The space of objects we're considering is the domain of $f(x)$: the set of real numbers $\mathbb{R}$. Then, the statement above is an existential quantifier that states there exists some real number a such that $f(a) = 0$.

Example 0.10.2. Every odd degree polynomial has a zero.

This is a universal quantifier with an existential quantifier inside. For the universal quantifier, the space of objects we're considering are all odd degree polynomials.

Example 0.10.3. **Mean Value Theorem**. Let $f : [a, b] \to \mathbb{R}$ be a function such that $f(x)$ is continuous on

$[a, b]$ and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Like the previous example, this is a universal quantifier with an existential quantifier inside. For the universal quantifier, we refer to the space of all functions that are continuous on $[a, b]$ and differentiable on (a, b).

Then, the existential quantifier considers the set of real numbers between a and b.

In this course, some statements may be expressed as implications but are understood as implications in universal quantifiers instead. We provide an example below.

Example 0.10.4. If $x = -2$, then $(x - 4)^2 = 0$.

This is a universal quantifier on the space of real numbers $\mathbb{R}$ with a implication predicate. This statement can be expressed equivalently as: for all $x \in \mathbb{R}$: if $x = -2$, then x satisfies $(x - 4)^2 = 0$ (i.e. the equation is true and we get $0 = 0$ when we plug in x).

While it may not make sense to understand this as a universal quantifier since the condition $x = -2$ uniquely identifies x, it matters when we look at $x = -2$ as an equation rather as an assignment. The next example covers why this matters.

Example 0.10.5. If $(x - 4)^2 = 0$, then $x = -2$.

This is a false statement. From algebra, we know that $(x - 4)^2 = 0$ has two solutions: $x = 2$ and $x = -2$. The implication statement above is equivalent to saying that if x satisfies $(x - 4)^2 = 0$, then x is uniquely identified to be $x = -2$ – which is false unless we are only considering x such that $x \leq 0$. Looking at it as a universal quantifier: $x = 2$ satisfies $(x - 4)^2 = 0$ but $x = 2$ does not satisfy $x = -2$.

A more correct statement would be: If $(x - 4)^2 = 0$, then $x = 2$ or $x = -2$. If we add the additional requirement of $x \leq 0$, we can conclude by disjunctive syllogism that $x = -2$.

Sometimes, we might come across a result that tells us that a quantifier (universal or existential) is false. To use those results, we provide negations of universal and existential quantifiers below. By the definition of negation, the negation of a predicate is true when the predicate is false.

Theorem 0.11. Negation of Universal and Existential Quantifiers.
Let $P(x)$ be a predicate. Then, the following bijections/equivalences apply:

1. $\textbf{NOT}\ \big(\forall\, x \in X, P(x)\big) \iff \exists\, x \in X,\ \textbf{NOT}\ P(x)$.
2. $\textbf{NOT}\ \big(\exists\, x \in X, P(x)\big) \iff \forall\, x \in X,\ \textbf{NOT}\ P(x)$.

Example 0.11.1. We then look at the earlier example: If $(x - 4)^2 = 0$, then $x = -2$. As earlier, we can understand this as: For all $x \in \mathbb{R}$: if x satisfies $(x - 4)^2 = 0$, then $x = -2$. We know this is false. We can make this statement true by negating it. For brevity, let $A(x)$ return true if x satisfies $(x - 4)^2 = 0$ and let $B(x)$ return true if $x = -2$. Then,

$$\begin{aligned} \textbf{NOT}\ \Big(\forall\, x \in \mathbb{R}, A(x) \Rightarrow B(x)\Big) &\iff \exists\, x \in \mathbb{R},\ \textbf{NOT}\ (A(x) \Rightarrow B(x)) \\ &\iff \exists\, x \in \mathbb{R}, (\textbf{NOT}\ B(x))\ \textbf{AND}\ A(x) \end{aligned}$$

In other words, there exists $x \in \mathbb{R}$ such that $x \neq -2$ and x satisfies $(x - 4)^2 = 0$. Recall that $x = 2$ makes this existential quantifier true.

Also, observe that universal quantifiers $\forall x \in X, P(x)$ can also be stated as implications $x \in X \Rightarrow P(x)$. Expressing the relationship this way is sometimes convenient when we want to use some theorems/results that give us the falsity of $P(x)$. Then, we can conclude that $x \notin X$ (read as x is not in X) by modus tollens.

Example 0.11.2. **Intermediate Value Theorem (IVT).** Let $f(x)$ be continuous on $[a, b]$. Then, for all y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$. For the sake of this example, let $I(f(x))$ be the predicate that accepts functions $f(x)$ and returns true if $f(x)$ satisfies IVT.

However, assuming the Intermediate Value Theorem is false for some function $g(x)$, then we can conclude that $g(x)$ is not continuous on $[a, b]$. In other words, we're using the contrapositive of this implication: If $g(x)$ is continuous on $[a, b]$, then $I(g(x))$ is true.

Going further, negating the universal quantifier on $y \in [f(a), f(b)]$ (assuming $f(a) \leq f(b)$), we can conclude that there exists y such that there is no $x \in [a, b]$ such that $f(x) = y$.

For this course, we would rarely determine the truth value of existential and universal quantifiers (i.e. if we're determining if those are true for some set of objects X, then we're actually doing proofs). Most of the time, we will provide those results for you and we'll use said results for our calculations.

Lastly, we introduce **uniqueness** and what that means logically.

Definition 0.12. Uniqueness.

Let X be some set of objects with a notion of equality denoted by $a = b$ for objects $a, b \in X$. Let $P(x)$ be a predicate on X. We say that x **uniquely** satisfies $P(x)$ if the universal quantifier $\forall x, y \in X : \big(P(x) \textbf{ AND } P(y)\big) \Rightarrow x = y$ is true.

Uniqueness is often a major result in mathematics and allows us to refer to an object as "the" object related to some property. In other words, there is exactly one object in X (distinguishable by that notion of equality) that satisfies a certain property.

Example 0.12.1. An additive inverse of a real number $x \in \mathbb{R}$ is y such that $x + y = 0$. It can be proven that this additive inverse is unique (i.e. there is only one real number y satisfying $x + y = 0$ for fixed x) and that it is given by $y = -x$. Therefore, we can speak of **the** additive inverse of a real number x.

That example may seem obvious but there are other cases where uniqueness is either not immediately obvious and hard to prove.

Example 0.12.2. Let $L = \{(x, y) \in \mathbb{R}^2\}$ be a line. Define m as the following expression:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

for any pairs of points (x_1, y_1) and (x_2, y_2). Then, m is uniquely determined (i.e. there is no other value k such that $k \neq m$ and k satisfies the expression above for any pairs of points). We call m **the** slope of the line. This is either given to you as a definition or as a result.

Example 0.12.3. Let x be a nonzero rational number. **The** reciprocal of x is a rational number y such that $x \cdot y = 1$.

Observe that if $x = \frac{1}{2}$, then $y = \frac{2}{1}$ and $y = \frac{4}{2}$ both satisfy $xy = 1$. However, because we consider $\frac{2}{1}$ and $\frac{4}{2}$ as equal as rational numbers, the reciprocal of x is unique.

Note that we've used the notion that two rational numbers $\frac{p}{q}, \frac{r}{s}$ are equal if and only if there exists a nonzero rational number x such that $p = xr$ and $q = xs$.

When the notion of equality used is not the usual "equality", when often say that it's "up to" that notion of equality.

Example 0.12.4. Given a plane P in $\mathbb{R}^3$, a normal vector $N \in \mathbb{R}$ is a vector such that for all $x, y \in P$, N is orthogonal to $x - y$. This normal vector is unique up to multiplication by a nonzero scalar.

In other words, if $N = (0, 2, 3)^T \in \mathbb{R}^3$ is a normal vector to P, the vector $2N = (0, 4, 6)^T$ is also a normal vector to P and so is $-3N = (0, -6, -9)^T$. Similarly, if some other vector $v \in \mathbb{R}^3$ is not a nonzero scalar multiple of N, then v is not a normal vector.

Finally, we introduce some terms that may appear in our discussion for this course. These terms have been used before this part of the document and are used in the same context. Do note that these terms may be understood differently in other contexts/areas and this is what the author has seen so far (i.e., up to the Master's level).

Convention 0.13.

Here are some commonly used terms in this course.

1. **Definition.** This is a placeholder name in the usual language that states some predicate $P(x)$ is true. This makes discussions of said objects easier to understand. While definitions are usually stated as implications, they're understood as bijections.
2. **Proposition.** A result stating the truth of some quantifier.
3. **Theorem.** This is major result stating the truth of some quantifier. Note here that the term "major" is relative, i.e. some theorems may prove results in a bigger set of objects than other theorems.

Some of these terms may also pop up in our discussion but will appear less often.

4. **Lemma.** This is a minor result, often used in a proof of a theorem with the same assumptions as the theorem in question. Again, the term "minor" is relative.
5. **Corollary.** This is a result that immediately follows a theorem. Here, immediately implies that a relatively small set of results and inference rules were used.
6. **Conjecture.** A claim stating the truth of some quantifier on a tentative basis without proof. i.e. we don't know *for certain* that it is true but it is *probably* true based on some reasoning.

Here are some examples of usage for those terms. Some of these will be discussed in MTH 264.

Example 0.13.1. **Definition.** A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$ where I_n is the identity matrix on $\mathbb{R}^{n\times n}$.

We use this in either of two ways: (1) we assume that a matrix A is invertible and say B is an inverse of A. Then, we use B in some calculation/proof; (2) we show that a matrix is invertible by showing the existential quantifier is true, i.e. we find B such that $AB = BA = I_n$. Later in the course, you'll see that we can find B by Gaussian elimination.

Example 0.13.2. **Theorem.** Fix $A \in \mathbb{R}^{n\times n}$ such that A is invertible. Let $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$. Then, B is unique.

This result is needed to define B as **the inverse** of A and to denote $A^{-1} = B$. Otherwise, the notation A^{-1} is ambiguous.

Example 0.13.3. **Corollary**. The inverse of the identity matrix I_n is itself, i.e. $I_n^{-1} = I_n$.

This immediately follows the theorem in the previous example since $I_n I_n = I_n$.

Example 0.13.4. **Fermat's Last Theorem**: there exists no three positive integers a, b, and c such that for all integers values n with $n > 2$, $a^n + b^n = c^n$.

Despite the naming, Fermat's Last Theorem, stated around 1637 by Pierre de Fermat, stayed as a conjecture for 358 years. Fermat stated that he had a proof but it was too large to fit in the margin. The first proper proof was finally published by Andrew Wiles in 1995.

Note that different definitions may be used across various fields or levels of generality. When a different definition is provided, there is usually some theorem/result that states that both definitions (the old and the new) agree in some specific sense.

Example 0.13.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The following are definitions for the continuity of $f(x)$ at $x = a \in \mathbb{R}$.

In differential calculus: A function $f(x)$ is continuous at $x = a$ if and only if $f(a) = \lim_{x\to a} f(x)$. Observe that the second statement requires that the limit $\lim_{x\to a} f(x)$ exists.

In topology/advanced calculus: A function $f(x)$ is continuous at $x = a$ if and only if for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $x \in \mathbb{R}$ satisfying $|x - a| < \delta_\epsilon$, $|f(x) - f(a)| < \epsilon$. Observe that (1) δ has the subscript of ϵ to denote that the value of δ depends on the given ϵ and (2) this definition looks very similar (but not the same) to the epsilon-delta definition of limits introduced in differential calculus.

Note: If you don't see how these two definitions agree, don't worry about it for now. In fact, we don't see this in this course. I've included this here since you might see alternate definitions for terms we've used in this course if you use other resources. That's fine.

Alternate definitions can be convenient for several reasons: e.g. (1) we need to generalize the current definition to a bigger set of objects; (2) the alternate definition is easier to use in proofs; (3) the alternate definition is easier to understand.

As a final note, it's not very critical that you use these terms right. However, some of the material in this class may be stated in the language above and so it's added here for reference.